Dual superfluid-Bose glass critical point in two dimensions and the universal conductivity

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Abstract: The continuum version of the dual theory for a system of two-dimensional quantum disordered bosons with statistical particle-hole symmetry and at $T = 0$ admits a new disordered critical point within the renormalization group calculation at fixed dimension. We obtain the universal conductivity and the critical exponents at the superfluid-Bose glass transition in the system: $\sigma_c = 0.25(2e)^2/h$, $\nu = 1.38$ and $z = 1.93$, to the lowest-order in fixed-point values of the dual coupling constants.

The problem of competition between superfluidity and localization in two dimensions (2D) has attracted lot of interest, as 2D is the lower critical dimension for both transitions when considered separately [1]. Experiments on disordered thin superconducting films show a sharp separation between samples of varying thickness that exhibit diverging or vanishing conductivities as temperature is lowered, providing support for the concept of quantum ($T = 0$) superconductor-insulator (SI) critical point [2]. Under the assumption that the amplitude of the superconducting order parameter remains finite at the critical value of the tuning parameter, it is sensible to describe the SI transition solely in terms of bosonic (phase) degrees of freedom in a random external potential. The resulting problem of disordered interacting bosons [3], [4] is interesting on its own account, and has direct implications for the phase transitions in other systems like ^4He in porous media, Josephson junctions arrays or granular superconductors. Not a small part of the fascination with the SI transition in 2D comes from a theoretical possibility that right at the transition point there is a finite dc conductivity, and moreover, that this conductivity in units of $(2e)^2/h$ should be a universal number [1]. The universal metallic behavior at the SI transition is an option unique for 2D, which posed a challenge for analytical calculations of the critical conductivity [5], particularly in the disordered case where this quantity has hitherto been obtained only numerically [6]. Generally, the critical behavior of disordered interacting bosons in more than one dimension [7] has proven very difficult to address analytically [8], [9], partly due to the fact that there is no easily identifiable correct mean-field starting point around which to set up a systematic perturbative calculation.

In this Letter we present the results of a new approach to the quantum critical behavior of 2D disordered bosonic systems, applied to the paradigmatic problem of disordered lattice bosons at a commensurate density, interacting via short-range repulsion. Without disorder and at $T = 0$ the system (described by the Hamiltonian (1)) undergoes a Mott insulator-superfluid (MI-SF) transition as the ratio t/U is varied, which is due to a commensurable boson density in the universality class of the 3D XY model [4]. With disorder the transition into a superfluid state is expected to always proceed from the gapless Bose glass (BG) phase [4]. We study the SI transition in this system within the continuum version of the *dual* theory for the bosonic Hamiltonian [10], which describes topological defects in the quantum ground state, and for the problem at hand takes the form of 3D classical anisotropic scalar electrodynamics (Ginzburg-Landau superconductor) in random magnetic field. The dual formulation of the problem may be thought of as a higher-dimensional analog of the Haldane's density representation of dirty bosons in 1D [7]. Since in dual representation the Gaussian point represents the *interacting* superfluid phase of the original bosons, one could expect that perturbing around it may be more sensible than in the original formulation, where one expands around the limit of non-interacting particles. Critical behavior of the isotropic version of the dual theory without disorder has been studied recently in different context by perturbative and non-perturbative methods [11], [12], [13]. While a small anisotropy at the critical point of the pure theory is *irrelevant*, the correlated random magnetic field is exactly *marginal* perturbation to the lowest order in disorder. However, we show that it generates a correlated random-mass term for the disorder field, which represents a strongly

relevant perturbation at the pure critical point. In one-loop approximation and for the physical initial values of the coupling constants the flow at the critical surface is attracted to a new disordered fixed point, which governs the critical behavior at the BG-SF transition in the system. We calculate the universal *resistivity* at the BG-SF transition, as well as the correlation length and the dynamical critical exponents ν and z, as series in fixed-point values of the dual coupling constants, to the lowest non-trivial order. The comparison with previous studies and with experiments is discussed.

Specifically, we are interested in the Hamiltonian for a collection of charge-2e bosons with short-range repulsion, $U > 0$, written in the number-phase representation [4], [10]:

$$\hat{H} = U \sum_i \hat{n}_i^2 - \sum_i h_i \hat{n}_i - t \sum_{\langle i,j \rangle} cos(\hat{\phi}_i - \hat{\phi}_j), \tag{1}$$

where index i labels the sites of a 2D lattice, $\hat{n}_i$ represents the deviations from a large *integer* average number of bosons per site, $\{h_i\}$ are Gaussian random variables with zero mean and width w_h, and $\hat{\phi}_i$ is the phase variable canonically conjugate to the number of bosons, $[\hat{\phi}_i, \hat{n}_j] = i\delta_{ij}$. The Josephson term couples only the nearest neighbours. Using a set of exact and nearly-exact transformations on a lattice [14], [15] Fisher and Lee [10] have demonstrated that the partition function for the disordered problem at $T = 0$ may be written in the form of classical 3D (two spatial and one imaginary time dimensions) strongly anisotropic lattice superconductor in a random magnetic field. The dual lattice theory should be in the same universality class as the continuum field theory for a strongly type-II superconductor ($\lambda >> q^2$) [15]:

$$\begin{aligned} S = \int d^2\vec{r}dz[|(\nabla - iq\vec{A}(\vec{r},z))\Psi(\vec{r},z)|^2 + \mu^2|\Psi(\vec{r},z)|^2 + \frac{\lambda}{2}|\Psi(\vec{r},z)|^4 \\ +\frac{1}{2}(\nabla \times A(\vec{r},z))^2 + \frac{\alpha}{2}(\nabla \times \vec{A}(\vec{r},z))^2_{\hat{z}} + h(\vec{r})(\nabla \times \vec{A}(\vec{r},z))_{\hat{z}}], \end{aligned} \tag{2}$$

where the complex Ψ represents the disorder-field [16]. The condensation of Ψ implies the proliferation of vortices in the ground state and consequently, the *destruction* of superfluidity by giving (Higgs) mass to the superfluids sound mode, represented by the dual gauge-field $\vec{A}$ ($\nabla \cdot \vec{A} = 0$). Note that the random magnetic field along $\hat{z}$ (time) direction $h(\vec{r})$ depends only on two (real space) coordinates, so that disorder is correlated along the third direction, as typical for quantum problems with quenched disorder. The condition on the average field $\overline{h} = 0$ corresponds to integer average number of particles per site.

The SI transition in dual theory is reached by tuning the renormalized mass of the disorder-field to zero. Introducing replicas in standard way to average over disorder, the dual theory at $m = 0$ may be parametrized by four dimensionless coupling constants: the dual charge $\hat{q}^2 = q^2/p$, the quartic term coupling $\hat{\lambda} = \lambda/p$, the anisotropy parameter α, and the coupling $\hat{w}_h = w_h/p$, which appears in the disorder-induced term of the form $-(w_h/2)\sum_{\alpha,\beta}\int(\nabla \times \vec{A}_\alpha(\vec{r},z))_{\hat{z}}(\nabla \times \vec{A}_\beta(\vec{r},z'))_{\hat{z}}$. p is an arbitrary infrared scale. Besides these, the random magnetic field *generates* yet another quartic coupling constant $\hat{w} = w/\pi p^2$ via the term $-(w/2)\sum_{\alpha,\beta}\int|\Psi_\alpha(\vec{r},z)|^2|\Psi_\beta(\vec{r},z')|^2$ in the replicated theory, with a meaning of

width of the Gaussian distribution of the random-mass term (again correlated along $\hat{z}$ axis) for the disorder-field. To determine the flow of the five coupling constants we performed a perturbative RG calculation in fixed dimension [11], [12], [13], since we will eventually be interested in the critical conductivity, which is finite only in 2D. The gist of the method is to express the universal quantities as series in renormalized, instead of bare, coupling constants, so they tend to finite values in the infrared limit, determined only by the location of the attractive fixed point. We define the renormalized couplings λ and w as the corresponding quartic vertices right at $m = 0$ and at the usual symmetric point in the momentum space: $\vec{k}_i \cdot \vec{k}_j = (4\delta_{i,j} - 1)p^2/4$, $i, j = 1, 2, 3$, with additional condition $k_{1,z} = k_{3,z} = -k_{2,z} = -k_{4,z}$ to accommodate the correlated nature of the disorder vertex w along z-axis. The polarization diagram which renormalizes q, α and w_h is evaluated at the external momentum $c \cdot p$, where the parameter c will be specified shortly. The β-functions for the anisotropy α and the random-field width w_h are:

$$\frac{d\alpha}{dt} = -\alpha(\frac{c}{16}\hat{q}^2 + O(\hat{q}^4)) + O(\alpha^2), \tag{3}$$

$$\frac{d\hat{w}_h}{dt} = \hat{w}_h(1 - \eta_A) + O(\hat{w}_h^2), \tag{4}$$

where $t = \ln(1/p) \to \infty$, in the infrared limit, and renormalized couplings are assumed. Eq. 3 is a one-loop result, and suggests that a small anisotropy is *irrelevant* at any fixed point with a finite dual charge. Although the dual theory should be strongly and not weakly anisotropic [10], we do not expect stable fixed points with $\alpha \neq 0$. This is known to be true in the classical scalar electrodynamics close to four dimensions and for a large number of complex field components [17]. In contrast to eq. 3, eq. 4 is *exact* to the lowest order in $\hat{w}_h$. This follows from the fact that the transverse part of polarization is diagonal in replica indices to the lowest order in $\hat{w}_h$, so that the renormalization of $\hat{w}_h$ to this order entirely comes from rescaling of the gauge-field. The anomalous dimension of the gauge-field propagator is $\eta_A = 1$ in 3D at any fixed point with a finite dual charge [12]. In particular, the linear term in eq. 4 vanishes at the pure MI-SF fixed point (see below). This observation may explain why numerical calculations at commensurate boson densities always have difficulties in detecting the intervening Bose glass phase at weak disorder [18]. It also seems consistent with the result of Singh and Rokshar [9], where a weak disorder in bosonic Hubbard model is found to be irrelevant, but only unusually weakly so. Fate of the coupling constant $\hat{w}_h$ is determined by the higher order terms in eq. 4. To proceed, we will *assume* that $\hat{w}_h$ is irrelevant at the MI-SF fixed point of the theory [19], and that its only role was to generate a strongly relevant coupling $\hat{w}$, before renormalizing to zero. Hereafter we will therefore set both $\alpha = \hat{w}_h = 0$.

One-loop β-functions for the remaining three coupling constants are then:

$$\frac{d\hat{q}^2}{dt} = \hat{q}^2 - \frac{c}{16}\hat{q}^4, \tag{5}$$

$$\frac{d\hat{\lambda}}{dt} = \hat{\lambda}(1 + \frac{1}{2}\hat{q}^2 + 2(\sqrt{2}\ln(1+\sqrt{2}) + \frac{1}{\sqrt{6}}\ln(\sqrt{2}+\sqrt{3}))\hat{w}) - \frac{2\sqrt{2}+1}{8}\hat{\lambda}^2 - \frac{1}{2\sqrt{2}}\hat{q}^4, \tag{6}$$

$$\frac{d\hat{w}}{dt} = \hat{w}(2 + \frac{1}{2}\hat{q}^2 - \frac{1}{2\sqrt{2}}\hat{\lambda}) + 2(\frac{1}{\sqrt{2}}\ln(1+\sqrt{2}) + \frac{1}{\sqrt{6}}\ln(\sqrt{2}+\sqrt{3}))\hat{w}^2. \tag{7}$$

For $\hat{w} = 0$ these equations reduce to those studied previously in the context of critical behavior of superconductors [12]. In that case for a choice $c > 5.17$ [20] there are four fixed points in the theory: Gaussian ($\hat{q}^2 = \hat{\lambda} = 0$) and 3D XY ($\hat{q}^2 = 0$, $\hat{\lambda}_{xy} = 2.09$), both unstable in direction of the dual charge, tricritical ($\hat{q}^2 = 16/c$, $\hat{\lambda} = \hat{\lambda}_-$), unstable in $\hat{\lambda}$-direction, and the MI-SF critical point ($\hat{q}^2 = 16/c$, $\hat{\lambda} = \hat{\lambda}_+$), believed to be of "inverted" XY type [14] (see Figure 2. in ref. 12). The Gaussian and the tricritical fixed points are connected by the straight separatrix determined by the tricritical value of the Ginzburg-Landau parameter $\kappa = \sqrt{\lambda/2q^2} = \kappa_c$ in the problem. On the other hand, the tricritical value of κ has been estimated both analytically [13] and numerically [21], which allows us to fix the parameter c to match that number: $\kappa_c = 0.42/\sqrt{2}$ requires $c = 20$. It is worth noting that for $\hat{w} = 0$ our calculation gives reasonable estimates of the critical exponents both at the unstable XY ($\nu = 0.63$) and at the stable MI-SF fixed point ($\nu = 0.61$, $\eta = -0.20$).

A small $\hat{w}$ is relevant at the MI-SF fixed point, as follows from eq. 7 and from the Harris criterion [22]. Both perturbative [12] and non-perturbative [13], [14], [15] calculations of the correlation length exponent at the pure critical point yield values $\nu < 1$, suggesting relevance of the correlated random-mass term. Starting from the type-II region ($\lambda >> q^2$), the flows are attracted by the disordered critical point at $\hat{q}_c^2 = 0.8$, $\hat{w}_c = 3.71$ and $\hat{\lambda}_c = 29.72$, which we identify as the BG-SF critical point. The BG-SF critical point exists for any choice of renormalization procedure (parameterized by c), unlike the charged fixed points in the pure theory which exist only for $c > 5.17$. This is akin to the classical scalar electrodynamics close to 4D [23], where randomness restores the critical point in the theory. To the lowest order in the critical coupling constants the exponents at the BG-SF fixed point are:

$$\nu = \frac{1}{2}(1 + \frac{\hat{\lambda}_c - \hat{q}_c^2 - 4\hat{w}_c}{8}) = 1.38, \tag{8}$$

$$z = 1 + \frac{\hat{w}_c}{4} = 1.93. \tag{9}$$

Note that $\nu > 1$, as expected [22]. The dynamical exponent is very close to two, which was conjectured to be an exact result [4] in 2D. The exponents are also in good agreement with Monte Carlo calculations of Wallin et al. [6] ($\nu = 0.9 \pm 0.1$, $z = 2 \pm 0.1$) and real-space study of Zhang and Ma [9] ($\nu = 1.4$, $z = 1.7$).

The disorder-field Ψ describes condensation of topological defects in the bosonic ground state, and duality of charges and vortices implies that the dimensionless conductance for vortices is the *resistance* for the original bosons. Kubo formula for the conductivity (or conductance, in 2D) of the vortices described by the dual theory then leads to the critical dc resistivity at $T = 0$:

$$R_c = \frac{2\pi h}{(2e)^2} \lim_{\omega\to 0} \frac{1}{\omega}[2\overline{\langle \Psi^*(0)\Psi(0)\rangle} - \int d^2\vec{r} dz e^{i\omega z} \overline{\langle j_x(\vec{r},z) j_x(\vec{0},0)\rangle}], \tag{10}$$

where $\vec{j}$ is the current of the dual field. To the lowest order we obtain

$$R_c = (\frac{\pi}{8} + \frac{\pi}{16}\hat{q}_c^2 + \frac{2.94}{\pi}\hat{w}_c)\frac{h}{(2e)^2}. \tag{11}$$

The first term describes the Gaussian part of the resistivity at the critical point [5], while the second term accounts for the lowest order wave-function renormalization. The third, disorder term, is the main source of dissipation. Inserting the values of the critical coupling constants into eq. 11 we estimate the universal conductivity for the bosons:

$$\sigma_c = R_c^{-1} = 0.25\frac{(2e)^2}{h}. \tag{12}$$

The result is somewhat larger than in Monte Carlo calculations of Wallin et al. [6]: $\sigma_c = [0.14 \pm 0.03](2e)^2/h$, and smaller than in Batrouni et al. [6]: $\sigma_c = [0.45 \pm 0.07](2e)^2/h$. Our preliminary calculations suggest that the higher order corrections tend to lower the value of σ_c. Remark however that the value of σ_c is already smaller than in the pure case [5].

The reader has certainly noted that the BG-SF fixed point in question is a strong-coupling one, and the numerical values of critical quantities cited above need to be taken with some reservation. There is no small parameter in the problem and we had to rely on a less controlled calculational scheme in fixed dimension [11]. Also, a different tricritical value of GL parameter κ_c in zero-disorder scalar electrodynamics would dictate a different choice of the parameter c, and would affect the position of the SF-BG fixed point within our calculation. The dependence of the critical quantities on this variation seems weak, fortunately. For instance, assuming almost a twice larger value for $\kappa_c = 0.8/\sqrt{2}$ [16] we would find $\nu = 1.55$, $z = 2.14$ and $\sigma_c = 0.18(2e)^2/h$. Our calculation allows for a systematic improvement by going to higher orders in the dual coupling constants and using standard resummation techniques [24]. Although the perturbation series in eqs. 3-7 are likely to be only asymptotic (but Borel summable [24]), it is encouraging that the lowest order calculation already produces sensible results.

We have observed that our numbers for the critical quantities are not too different from those obtained by other methods performed on the Hamiltonian (1) [6] or on its variant [9], but at a *non-commensurate* density of bosons. Without disorder, at non-commensurate filling factors the MI-SF transition is in a different universality class, and it is in fact mean-field in character [4]. One would expect on physical grounds that commensurability may be irrelevant at the disordered critical point [8]. Deviation from a commensurate density would correspond to a finite average magnetic field $\overline{h}$ in dual theory, whose scaling dimension at the BG-SF fixed point we do not presently know. We have argued however, that even at commensurate densities the transition is always controlled by the disordered BG-SF fixed point, due to the generated random-mass term for the disorder field.

Comparison with experiments is made difficult by the fact that different measurements lead to rather different values of the three critical quantities we considered [2]. We may still note that our numbers for the critical exponents fall within the range $\nu z = 2.8 \pm 0.4$ obtained from the scaling of resistivity on the insulating side of the SI transition in thin Bi

films [2]. The value of the critical conductivity in thin films differs by a factor of ~ 4 in various experiments [2], and its universality may be questioned. Many experiments show $\sigma_c \sim (2e)^2/h$ [2], somewhat larger than our estimate. Note also that experimentally one typically measures $\omega = 0$, $T \to 0$ conductivity [7], which is the opposite order of limits than assumed here. It has been argued recently that the two limits may not commute [25], and that the critical conductivity has a non-trivial structure as a function of ω/T.

In conclusion, we considered the critical behavior at the superfluid-insulator transition in a system of 2D disordered bosons with commensurate density and at $T = 0$, by studying the continuum version of the dual theory that describes vortices in the superfluid ground state. Although disorder, which initially appears in the dual theory as random magnetic field, is argued to be marginally irrelevant at the critical point of the pure theory, it generates another strongly relevant disorder-like coupling constant, and the transition is ultimately governed by the BG-SF critical point. The universal resistivity at the BG-SF transition and the critical exponents have been expressed as perturbation series in the critical coupling constants, and the lowest order results compare well with the Monte Carlo calculations performed on the original boson Hamiltonian.

The author is thankful to Professors I. Affleck, P. Stamp and Z. Tešanović for useful discussions. This work has been supported by NSERC of Canada and Izaak Walton Killam foundation.

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